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Quasiconvex stochastic processes and a separation theorem

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Dedicated to Professor János Aczél on his 90^{th} *birthday*

Abstract. Quasiconvex stochastic processes are introduced. A characterization of pairs of stochastic processes that can be separated by a quasiconvex stochastic process and a stability theorem for quasiconvex processes are given.

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1. Introduction

In [1], Baron et al. proved that two real functions f and g defined on a real interval I can be separated by a convex function if and only if they fulfil the following inequality

$$f\big(\lambda x+(1-\lambda)y\big)\leqslant \lambda g(x)+(1-\lambda)g(y),$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

In 1994 Smolarz [11] obtained an analogous result for quasiconvex functions. Namely, he proved that two functions $f, g : I \to \mathbb{R}$ can be separated by a quasiconvex function if and only if

$$f\big(\lambda x+(1-\lambda)y\big)\leqslant \max\big\{g(x),g(y)\big\},$$

for all $x, y \in I$ and $\lambda \in [0,1]$ (see also [3]).

In this paper we introduce the notion of quasiconvex stochastic processes and present some properties of them. In particular we show that a stochastic process is convex if and only if it is Jensen-convex and quasiconvex. Our main result extends the Smolarz separation theorem to quasiconvex stochastic processes. As a consequence we obtain a Hyers-Ulam-type stability result for quasiconvex stochastic processes.

2. Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space and $I \subset \mathbb{R}$ be an interval. A function $X : \Omega \to \mathbb{R}$ is called a *random variable*, if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$ is called a *stochastic process*, if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Recall that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be *convex*, if

$$X\bigl(\lambda t_1 + (1-\lambda)t_2, \cdot\bigr) \leqslant \lambda X(t_1, \cdot) + (1-\lambda)X(t_2, \cdot) \quad \text{(a.e.)},$$

for all $t_1, t_2 \in I$ and $\lambda \in [0, 1]$.

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called *Jensen-convex*, if

$$X\Bigl(\frac{t_1 + t_2}{2}, \cdot\Bigr) \leqslant \frac{X(t_1, \cdot) + X(t_2, \cdot)}{2} \quad \text{(a.e.)},$$

for all $t_1, t_2 \in I$.

Convex and Jensen-convex stochastic processes were investigated by many authors and various properties and applications of them can be found in the literature (see, for instance, [5,6,9,10] and the references therein).

We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is *quasiconvex*, if

$$X\bigl(\lambda t_1 + (1-\lambda)t_2, \cdot\bigr) \leqslant \max\bigl\{X(t_1, \cdot), X(t_2, \cdot)\bigr\} \quad \text{(a.e.)}, \tag{1}$$

for all $t_1, t_2 \in I$ and $\lambda \in [0, 1]$.

We start our investigation with two simple observations.

Observation 1. *The following conditions are equivalent:*

(i) $X : I \times \Omega \to \mathbb{R}$ *is a quasiconvex stochastic process.*

(ii) *For every random variable* $A : \Omega \to \mathbb{R}$ *the level set*

$$L_A = \bigl\{t \in I : \; X(t, \cdot) \leqslant A(\cdot) \quad \textit{(a.e.)}\bigr\}$$

is convex.

Proof. Suppose first that X is a quasiconvex stochastic process and fix a random variable $A : \Omega \to \mathbb{R}$. Let $t_1, t_2 \in L_A$ and $\lambda \in [0, 1]$. By (1) and the definition of level sets we have

$$\begin{aligned} X\bigl(\lambda t_1 + (1-\lambda)t_2, \cdot\bigr) &\leqslant \max\bigl\{X(t_1, \cdot), X(t_2, \cdot)\bigr\} \\ &\leqslant \max\bigl\{A(\cdot), A(\cdot)\bigr\} = A(\cdot) \quad \text{(a.e.)}. \end{aligned}$$

Thus $\lambda t_1 + (1-\lambda)t_2 \in L_A$, which proves that L_A is convex.

Assume now that the sets L_A are convex for all random variables A. Fix $t_1, t_2 \in I$ and $\lambda \in [0, 1]$. Define $A(\cdot) = \max\bigl\{X(t_1, \cdot), X(t_2, \cdot)\bigr\}$. Then, of course,

$t_1, t_2 \in L_A$, and, by the convexity of L_A, we have $\lambda t_1 + (1-\lambda)t_2 \in L_A$. It means that the inequality

$$X\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) \leqslant A(\cdot) = \max\big\{X(t_1,\cdot), X(t_2,\cdot)\big\} \quad \text{(a.e.)}$$

holds and X is quasiconvex. □

Observation 2. *If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is convex, then it is quasiconvex.*

Proof. By the convexity of X, for all $t_1, t_2 \in I$ and $\lambda \in [0,1]$, we have

$$\begin{aligned} X\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) &\leqslant \lambda X(t_1,\cdot) + (1-\lambda)X(t_2,\cdot) \\ &\leqslant \max\big\{X(t_1,\cdot), X(t_2,\cdot)\big\} \quad \text{(a.e.)}, \end{aligned}$$

which shows that the process X is quasiconvex. □

Clearly, quasiconvex (as well as Jensen-convex) stochastic processes need not be convex. However, if a stochastic proces is both quasiconvex and Jensen-convex, then it is convex.

Proposition 3. *Let I be an open interval. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is convex if and only if it is quasiconvex and Jensen-convex.*

Proof. The "only if" part is clear. To prove the "if" part fix $t_1, t_2 \in I$, $t_1 < t_2$. By the quasiconvexity of X, for every $t \in [t_1, t_2]$, we have

$$X(t,\cdot) \leqslant \max\big\{X(t_1,\cdot), X(t_2,\cdot)\big\} \quad \text{(a.e.)}.$$

This implies that the process X is P-upper bounded on $[t_1, t_2]$, that is

$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} \{P(\{\omega \in \Omega : |X(t,\omega)| \geq n\})\} = 0.$$

Since X is also Jensen-convex, it follows, by the Bernstein-Doetsch-type theorem, that X is continuous in probability and, consequently, convex (see [6] Theorems 4, 5). □

3. Main result

At the beginning of this section we would like to recall the definition and basic properties of the essential infimum of a collection of functions. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{S}$ be a collection of measurable functions $f : \Omega \to \mathbb{R}$. On $\mathbb{R}$ the Borel σ-algebra is used. If $\mathcal{S}$ is a countable set, then we may define the pointwise infimum of the functions from $\mathcal{S}$, which will itself be measurable. If $\mathcal{S}$ is uncountable, then the pointwise infimum need not be measurable. In this case, the essential infimum can be used. The *essential infimum* of $\mathcal{S}$, written as ess inf $\mathcal{S}$, if it exists, is a measurable function $f : \Omega \to \mathbb{R}$ satisfying the two following axioms:

- $f \leqslant g$ almost everywhere, for any $g \in \mathcal{S}$,

- if $h : \Omega \to \mathbb{R}$ is measurable and $h \leqslant g$ almost everywhere for every $g \in \mathcal{S}$, then $h \leqslant f$ almost everywhere.

Note that if f is the essential infimum and $g : \Omega \to \mathbb{R}$ is equal to f almost everywhere, then g is also an essential infimum. Conversely, if f and g are both essential infima, then, from the above definition $f \leqslant g$ and $g \leqslant f$, so $f = g$ almost everywhere. It can be shown that for a σ-finite measure μ, the essential infimum of $\mathcal{S}$ does exist. Furthermore, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that

$$\operatorname{ess\,inf} \mathcal{S} = \inf\big\{f_n : n \in \mathbb{N}\big\}.$$

For more details we refer the reader to [2].

The following properties of essential infimum will be useful in the sequel.

Lemma 4. *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, $\mathcal{S}$ be a nonempty collection of measurable real functions defined on Ω, and let $g : \Omega \to \mathbb{R}$ be a measurable function. If* $\operatorname{ess\,inf} \mathcal{S} < g$ *almost everywhere, then there exist sets $\Omega_n \in \mathcal{F}$ and functions $f_n \in \mathcal{S}$ for $n \in \mathbb{N}$, such that $\mu\big(\Omega\setminus \bigcup_{n\in\mathbb{N}} \Omega_n\big) = 0$ and $f_n < g$ on $\Omega_n, n \in \mathbb{N}$.*

Proof. By the fact mentioned above there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{S}$ such that

$$\operatorname{ess\,inf} \mathcal{S} = \inf\big\{f_n : n \in \mathbb{N}\big\}.$$

Assume that $\inf\big\{f_n : n \in \mathbb{N}\big\} < g$ on $\overline{\Omega} = \Omega\setminus\Omega_0$, where $\mu\big(\Omega_0\big) = 0$. By the definition of the infimum, for every $\omega \in \overline{\Omega}$ there exists $n \in \mathbb{N}$ such that $f_n(\omega) < g(\omega)$.

Define the sets

$$\Omega_n = \big\{\omega \in \overline{\Omega} :\ f_n(\omega) < g(\omega)\big\}.$$

Then $\Omega_n \in \mathcal{F}$, $\bigcup_{n\in\mathbb{N}} \Omega_n = \overline{\Omega}$, and $f_n < g$ on Ω_n for $n \in \mathbb{N}$, which was to be proved. □

Lemma 5. *Let $f : \Omega \to \mathbb{R}$ be a measurable function and $\mathcal{S}$ be a family of all measurable functions $g : \Omega \to \mathbb{R}$ such that $f < g$ almost everywhere. Then* $\operatorname{ess\,inf} \mathcal{S} = f$.

Proof. Clearly $\operatorname{ess\,inf} \mathcal{S} \geqslant f$ almost everywhere. To conclude the proof it is enough to observe that $f + \frac{1}{n} \in \mathcal{S}$ for all $n \in \mathbb{N}$. □

Now we present our main theorem. It gives a condition under which two stochastic processes can be separated by a quasiconvex stochastic process.

Theorem 6. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be stochastic processes. There exists a quasiconvex stochastic process $H : I \times \Omega \to \mathbb{R}$ such that*

$$X(t, \cdot) \leqslant H(t, \cdot) \leqslant Y(t, \cdot) \quad \textit{(a.e.)}, \quad t \in I,$$

if and only if

$$X\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) \leqslant \max\big\{Y(t_1,\cdot), Y(t_2,\cdot)\big\} \quad \textit{(a.e.)} \tag{2}$$

for all $t_1, t_2 \in I$ *and* $\lambda \in [0,1]$.

Proof. The sufficiency is obvious. To prove the necessity assume that X and Y fulfil (2). Given a random variable $A : \Omega \to \mathbb{R}$ consider the level set

$$L_A = \big\{t \in I : \, Y(t,\cdot) \leqslant A(\cdot) \quad \text{(a.e.)}\big\}.$$

Let $C_A = \operatorname{conv} L_A$ denote the convex hull of the set L_A. Define a stochastic process $H : I \times \Omega \to \mathbb{R}$ by

$$H\big(t,\cdot\big) = \operatorname{ess\,inf}\big\{A : \, t \in C_A\big\}.$$

Fix $t_0 \in I$ and take a random variable $A : \Omega \to \mathbb{R}$ such that $t_0 \in C_A$. In view of the Caratheodory theorem (cf. [8]) we have $t_0 = \lambda t_1 + (1-\lambda)t_2$, for some $t_1, t_2 \in L_A$ and $\lambda \in [0,1]$. Hence $Y\big(t_1,\cdot\big) \leqslant A(\cdot) \quad \text{(a.e.)}$ and $Y\big(t_2,\cdot\big) \leqslant A(\cdot) \quad \text{(a.e.)}$ and, by inequality (2), we obtain

$$X\big(t_0,\cdot\big) = X\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant \max\big\{Y(t_1,\cdot), Y(t_2,\cdot)\big\} \leqslant A(\cdot) \quad \text{(a.e.)}.$$

Since the above inequality holds for any random variable A, such that $t_0 \in C_A$, we get

$$X\big(t_0,\cdot\big) \leqslant \operatorname{ess\,inf}\big\{A : \, t_0 \in C_A\big\} = H\big(t_0,\cdot\big) \quad \text{(a.e.)}.$$

Moreover, since for every fixed $t_0 \in I$ we have $t_0 \in L_{Y(t_0,\cdot)} \subset C_{Y(t_0,\cdot)}$, we get also

$$H\big(t_0,\cdot\big) = \operatorname{ess\,inf}\big\{A : \quad t_0 \in C_A\big\} \leqslant Y\big(t_0,\cdot\big) \quad \text{(a.e.)}.$$

Now we will show that the stochastic process H is quasiconvex. Fix $t_1, t_2 \in I$ and $\lambda \in [0,1]$. The following cases are possible:

(i) $H\big(t_1,\cdot\big) \leqslant H\big(t_2,\cdot\big) \quad \text{(a.e.)}$.
(ii) $H\big(t_1,\cdot\big) > H\big(t_2,\cdot\big) \quad \text{(a.e.)}$.
(iii) The following sets Ω_1 and Ω_2 have positive measure

$$\Omega_1 = \big\{\omega : \, H\big(t_1,\omega\big) \leqslant H\big(t_2,\omega\big)\big\},$$
$$\Omega_2 = \big\{\omega : \, H\big(t_1,\omega\big) > H\big(t_2,\omega\big)\big\}.$$

Assume first that case (i) holds. We will show that there exist sets $\Omega_{n,k} \in \mathcal{A}, n,k \in \mathbb{N}$, such that $P\big(\bigcup_{n,k\in\mathbb{N}} \Omega_{n,k}\big) = 1$ and for all $n,k \in \mathbb{N}, H\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant H\big(t_2,\cdot\big)$ almost everywhere on $\Omega_{n,k}$.
Take an arbitrary random variable $B : \Omega \to \mathbb{R}$ satisfying

$$H\big(t_2,\cdot\big) = \operatorname{ess\,inf}\big\{A : \, t_2 \in C_A\big\} < B(\cdot) \quad \text{(a.e.)}. \tag{3}$$

By Lemma 4 there exist sets $\Omega_n \in \mathcal{A}$ and random variables $A_n \in \{A : t_2 \in C_A\}$ such that $P\big(\bigcup_{n\in\mathbb{N}} \Omega_n\big) = 1$ and for every $n \in \mathbb{N}$, $A_n < B$ on Ω_n. Then

$$\begin{aligned} t_2 \in C_{A_n} &= \operatorname{conv}\big\{t \in I : Y\big(t,\cdot\big) \leqslant A_n(\cdot) \quad \text{(a.e.)}\big\} \\ &\subset \operatorname{conv}\big\{t \in I : Y\big(t,\cdot\big) < B(\cdot) \quad \text{(a.e.)} \quad \text{on} \quad \Omega_n\big\}. \end{aligned} \tag{4}$$

Now, we use the fact that $H\big(t_1,\cdot\big) \leqslant H\big(t_2,\cdot\big) < B\big(\cdot\big)$ (a.e.) and apply Lemma 4 separately for every set Ω_n, $n \in \mathbb{N}$. There exist sets $\Omega_{n,k} \subset \Omega_n$, $\Omega_{n,k} \in \mathcal{A}$ and random variables $A_{n,k} \in \big\{A : t_1 \in C_{A|_{\Omega_n}}\big\}$ such that $P\big(\bigcup_{k\in\mathbb{N}} \Omega_{n,k}\big) = P\big(\Omega_n\big)$ and $A_{n,k} < B$ on $\Omega_{n,k}$ for $n, k \in \mathbb{N}$. Then

$$\begin{aligned} t_1 \in C_{A_{n,k}|\Omega_n} &= \operatorname{conv}\big\{t \in I : Y\big(t,\cdot\big) \leqslant A_{n,k}(\cdot) \quad \text{(a.e.)} \quad \text{on} \quad \Omega_n\big\} \\ &\subset \operatorname{conv}\big\{t \in I : Y\big(t,\cdot\big) < B(\cdot) \quad \text{(a.e.)} \quad \text{on} \quad \Omega_{n,k}\big\}. \end{aligned} \tag{5}$$

By (4) and (5), for every $n, k \in \mathbb{N}$ we obtain

$$t_1, t_2 \in \operatorname{conv}\big\{t \in I : Y\big(t,\cdot\big) < B(\cdot) \quad \text{(a.e.)} \quad \text{on} \quad \Omega_{n,k}\big\}$$

and consequently

$$\lambda t_1 + (1-\lambda)t_2 \in \operatorname{conv}\big\{t \in I : Y\big(t,\cdot\big) < B(\cdot) \quad \text{(a.e.)} \quad \text{on} \quad \Omega_{n,k}\big\}.$$

Hence, by the definition of H, we get $H\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant B\big(\cdot\big)$ (a.e.) on $\Omega_{n,k}$. Since the family $\big(\Omega_{n,k}\big)_{n,k\in\mathbb{N}}$ is countable and $P\big(\bigcup_{k\in\mathbb{N}} \Omega_{n,k}\big) = P\big(\Omega_n\big)$ for $n \in \mathbb{N}$ and $P\big(\bigcup_{n\in\mathbb{N}} \Omega_n\big) = 1$, we also have $H\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant B\big(\cdot\big)$ (a.e.) on Ω. Using the fact, that this inequality holds for every random variable B satisfying (3), by Lemma 5 we obtain

$$H\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant H\big(t_2,\cdot\big) = \max\big\{H(t_1,\cdot), H(t_2,\cdot)\big\} \quad \text{(a.e.)}.$$

This finishes the the proof in case (i).
The proof in case (ii) is analogous, so we omit it.
Now assume that case (iii) holds. We consider two processes H_1 and H_2 being the restrictions of H to $I \times \Omega_1$ and $I \times \Omega_2$, respectively. Then for H_1 case (i) occurs and for H_2 case (ii) occurs. It means that

$$H_1\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant \max\big\{H_1(t_1,\cdot), H_1(t_2,\cdot)\big\} \quad \text{(a.e.)} \quad \text{on} \quad \Omega_1$$

and

$$H_2\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant \max\big\{H_2(t_1,\cdot), H_2(t_2,\cdot)\big\} \quad \text{(a.e.)} \quad \text{on} \quad \Omega_2.$$

Consequently

$$H\big(\lambda t_1 + (1-\lambda)t_2,\cdot\big) \leqslant \max\big\{H(t_1,\cdot), H(t_2,\cdot)\big\} \quad \text{(a.e.)} \quad \text{on} \quad \Omega$$

and the proof is complete. □

4. Hyers–Ulam stability

As a direct consequence of Theorem 6 we obtain the following Hyers–Ulam-type stability result for quasiconvex stochastic processes. For the classical Hyers–Ulam theorem see [4]. The stability theorem for quasiconvex functions was obtained in [7] (cf. also [11]).

Theorem 7. *Let ε be a positive constant. If a stochastic process $X : I \times \Omega \to \mathbb{R}$ satisfies the inequality*

$$X\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) \leqslant \max\big\{X(t_1,\cdot), X(t_2,\cdot)\big\} + \varepsilon \quad \textit{(a.e.)}$$

for all $t_1, t_2 \in I$ and $\lambda \in [0,1]$, then there exists a quasiconvex stochastic process $H : I \times \Omega \to \mathbb{R}$ such that $\big|X(t,\cdot) - H(t,\cdot)\big| \leqslant \frac{\varepsilon}{2}$ (a.e.) for every $t \in I$.

Proof. To prove the above theorem it is enough to apply Theorem 6 to the stochastic processes X and $X + \varepsilon$. Hence, there exists a process $H_1 : I \times \Omega \to \mathbb{R}$, which is quasiconvex and satisfies $X(t,\cdot) \leqslant H_1(t,\cdot) \leqslant X(t,\cdot) + \varepsilon$ (a.e.). By taking $H(t,\cdot) = H_1(t,\cdot) - \frac{\varepsilon}{2}$ we get $\big|X(t,\cdot) - H(t,\cdot)\big| \leqslant \frac{\varepsilon}{2}$ (a.e.). This completes the proof. □

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